TOPOLOGY - III, EXERCISE SHEET 13

Exercise 1. *Cellular homology of some important examples.*
Using the CW complex structures you came up with for the following spaces from exercise 1 of sheet 11, compute the corresponding cellular complexes and cellular homology groups:

(1) S^n.
(2) $(T^2)^{\#n}$
(3) $(\mathbb{RP}^2)^{\#n}$
(4) $\mathbb{RP}^n$
(5) $\mathbb{CP}^n$.
(6) Connected graphs, in terms of number of edges and vertices.

Exercise 2. *Homology of complex Grassmannians and Young Diagrams.*
The goal of this exercise is to demonstrate how the homology of complex Grassmannians can be calculated using the combinatorics of Young diagrams.

(1) Recall that the Grassmanian of $k-$planes in $\mathbb{C}^n$, denoted by $Gr(k,n)$ is the space parameterising dimension k subspaces of the complex vector space $\mathbb{C}^n$. Review from the last exercise sheet that we endowed $Gr(k,n)$ with a topology by realising it as a quotient of the space $V_k(\mathbb{C}^n)$.
Show that we can equivalently realise $Gr(k,n)$ as the quotient

$$Mat^{\circ}_{k\times n}(\mathbb{C})/GL_k(\mathbb{C}).$$

Where $Mat^{\circ}_{k\times n}(\mathbb{C})$ denotes the open subset of $k\times n$ complex matrices consisting of full-rank matrices and is acted on by $GL_k(\mathbb{C})$ via left multiplication or equivalently via row operations.

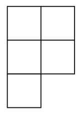
(2) Recall that a Schubert symbol is a sequence $\sigma = 1 \leq j_1 < j_2 < ... < j_k \leq n$. To such a Schubert symbol we associate the subset $W^{\circ}_{\sigma} \subseteq Mat^{\circ}_{k\times n}(\mathbb{C})$ consisting of those matrices whose j_m-th column is the column matrix $\begin{bmatrix} 0 \\ . \\ 1 \\ 0 \\ . \\ 0 \end{bmatrix}$, with the 1 in the $k-m+1$-th row and all the matrix entries in the $k-m+1$-th row, which are to the right of the 1 are equal to zero, ie. That is the $(k-m+1,l)-$th entry of the matrix is zero for $l > j_m$. As an example, for $Gr(3,5)$ and the Schubert symbol $\sigma = (2,4,5)$ we have that W°_{σ} consists of the matrices of the form

$$\begin{pmatrix} * & 0 & * & 0 & 1 \\ * & 0 & * & 1 & 0 \\ * & 1 & 0 & 0 & 0. \end{pmatrix}.$$

Where the $*$ denote arbitrary complex numbers. Denote by X^0_σ the image of W°_σ in $Gr(k,n) = Mat^\circ_{k\times n}(\mathbb{C})/GL_k(\mathbb{C})$.

(a) Show that X^0_σ is equal to the Schubert cell $e(\sigma)$ defined in sheet 12.
(b) Observe that the dim $X^0_\sigma = 2(j_1 + ... + j_k - k(k+1)/2)$, that is the number of $*$.
(c) List all Schubert symbols and the corresponding Schubert cells in terms of their matrix representation for $Gr(2,4)$ and $Gr(3,5)$.

(3) Given a positive integer j, a partition of j is a sequence $l_1 \geq l_2 \geq ... \geq l_r$ of positive integers such that $\sum_{i=1}^r l_i = j$. To the data of a partition of j we can associate a so-called Young Diagram, an arrangement of j boxes such that the number of boxes in the $i-$th row is l_i. For example the young diagram corresponding to the partition $(2,2,1)$ of 5 is given by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Note that to a Schubert cell, one can also associate a young diagram by counting the number of $*$ in each row of the matrix representation of the Schubert cell. For example the above Young diagram is associated to the Schubert cell in $Gr(3,5)$ defined by the Schubert symbol $(2,4,5)$.

It was shown in the last exercise sheet that the Schubert cells can be be used to define a CW space structure on $Gr(k,n)$. Since the Schubert cells are all of even dimension, we have that the boundary maps of the cellular complex for $Gr(k,n)$ are all zero and hence $H^{2j+1}(Gr(k,n)) = 0$ and $H^{2j}(Gr(k,n))$ is equal to the number of Schubert cells of dimension $2j$.

Show that $H^{2j}(Gr(k,n))$ is equal to the number of young diagrams of size j in a $k\times(n-k)$ grid.

Using this, compute the homology groups of the following:
(a) Gr(1,4).
(b) Gr(3,4).
(c) Gr(2,4).
(d) Gr(3,5).
(e) Gr(2,6).